

# The strong metric dimension of some generalized Petersen graphs \*

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In this paper the strong metric dimension of generalized Petersen graphs  $GP(n, 2)$  is considered. The exact value is determined for cases  $n = 4k$  and  $n = 4k + 2$ , while for  $n = 4k + 1$  an upper bound of the strong metric dimension is presented.

**Keywords:** Strong metric dimension, generalized Petersen graphs

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## 1 Introduction

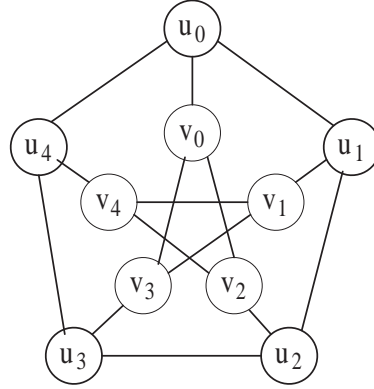
The strong metric dimension problem was introduced by Sebo and Tannier (2004). This problem is defined in the following way. Given a simple connected undirected graph  $G = (V, E)$ , where  $V = \{1, 2, \dots, n\}$ ,  $|E| = m$  and  $d(u, v)$  denotes the distance between vertices  $u$  and  $v$ , i.e. the length of a shortest  $u - v$  path. A vertex  $w$  strongly resolves two vertices  $u$  and  $v$  if  $u$  belongs to a shortest  $v - w$  path or  $v$  belongs to a shortest  $u - w$  path. A vertex set  $S$  of  $G$  is a *strong resolving set* of  $G$  if every two distinct vertices of  $G$  are strongly resolved by some vertex of  $S$ . A *strong metric basis* of  $G$  is a strong resolving set of the minimum cardinality. The *strong metric dimension* of  $G$ , denoted by  $sdim(G)$ , is defined as the cardinality of its strong metric basis. Now, the strong metric dimension problem is defined as the problem of finding the strong metric dimension of graph  $G$ .

**Example 1** Consider the Petersen graph  $G$  given on Figure 1. It is easy to see that set  $S = \{u_0, u_1, u_2, u_3, v_0, v_1, v_2, v_3\}$  is a strong resolving set, i.e. each pair of vertices in  $G$  is strongly resolved by a vertex from  $S$ . Since any pair with at least one vertex in  $S$  is strongly resolved by that vertex, the only interesting case is pair  $u_4, v_4$ . This pair is strongly resolved e.g. by  $u_0 \in S$  since the shortest path  $u_0, u_4, v_4$  contains  $u_4$ . In Section 2 it will be demonstrated that  $S$  is a strong resolving set with the minimum cardinality and, therefore,  $sdim(G) = 8$ .

The strong metric dimension has many interesting theoretical properties. If  $S$  is a strong resolving set of  $G$ , then the matrix of distances from all vertices from  $V$  to all vertices from  $S$  uniquely determines graph  $G$  (Sebo and Tannier, 2004).

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**Fig. 1:** Petersen graph  $G$

The strong metric dimension problem is NP-hard in general case (Oellermann and Peters-Fransen, 2007). Nevertheless, for some classes of graphs the strong metric dimension problem can be solved in polynomial time. For example, in (May and Oellermann, 2011) an algorithm for finding the strong metric dimension of distance hereditary graphs with  $O(|V| \cdot |E|)$  complexity is presented.

In (Kratica et al., 2012b) an integer linear programming (ILP) formulation of the strong metric dimension problem was proposed. Let variable  $y_i$  determine whether vertex  $i$  belongs to a strong resolving set  $S$  or not, i.e.  $y_i = \begin{cases} 1, & i \in S \\ 0, & i \notin S \end{cases}$ . Now, the ILP model of the strong metric dimension problem is given by (1)-(3).

$$\min \sum_{i=1}^n y_i \tag{1}$$

subject to:

$$\sum_{i=1}^n A_{(u,v),i} \cdot y_i \geq 1 \quad 1 \leq u < v \leq n \tag{2}$$

$$y_i \in \{0, 1\} \quad 1 \leq i \leq n \tag{3}$$

$$\text{where } A_{(u,v),i} = \begin{cases} 1, & d(u,i) = d(u,v) + d(v,i) \\ 1, & d(v,i) = d(v,u) + d(u,i) \\ 0, & \text{otherwise} \end{cases}$$

This ILP formulation will be used in Section 2 for finding the strong metric dimension of generalized Petersen graphs in singular cases, when dimensions are small. The following definition and two properties from literature will also be used in Section 2.

**Definition 1** (Kratika et al., 2012b) A pair of vertices  $u, v \in V$ ,  $u \neq v$ , is mutually maximally distant if and only if

- (i)  $d(w, v) \leq d(u, v)$  for each  $w$  such that  $\{w, u\} \in E$  and
- (ii)  $d(u, w) \leq d(u, v)$  for each  $w$  such that  $\{v, w\} \in E$ .

**Property 1** (Kratika et al., 2012b) If  $S \subset V$  is a strong resolving set of graph  $G$ , then, for every two maximally distant vertices  $u, v \in V$ , it must be  $u \in S$  or  $v \in S$ .

Let  $\text{Diam}(G)$  denote the diameter of graph  $G$ , i.e. the maximal distance between two vertices in  $G$ .

**Property 2** (Kratika et al., 2012b) If  $S \subset V$  is a strong resolving set of graph  $G$ , then, for every two vertices  $u, v \in V$  such that  $d(u, v) = \text{Diam}(G)$ , it must be  $u \in S$  or  $v \in S$ .

A survey paper (Kratika et al., 2014) contains the results related to the strong metric dimension of graphs up to mid 2013. Here we give a short overview of the newest results which are not covered in (Kratika et al., 2014):

- In (DasGupta and Mobasheri, 2014) an 2-approximation algorithm and a  $(2 - \epsilon)$ -inapproximability result for the strong metric dimension problem is given;
- In (Yi, 2013) a Nordhaus-Gaddum-type upper and lower bound for the strong metric dimension of a graph and its complement is given. The characterization of the cases when the bounds are attained is also presented;
- A comparison between the zero forcing number and the strong metric dimension of graphs is presented by Kang and Yi (2014);
- Closed formulas for the strong metric dimension of several families of the Cartesian product of graphs and the direct product of graphs are given in (Rodriguez-Velázquez et al., 2013; Kuziak et al., 2014a);
- In (Kuziak et al., 2014a, 2013) the authors study the strong metric dimension of the rooted product of graphs and express it in terms of invariants of the factor graphs;
- Several relationships between the strong metric dimension of the lexicographic product of graphs and the strong metric dimension of its factor graphs are obtained in (Kuziak et al., 2014b);
- The strong metric dimension of the strong product of graphs is analyzed in (Kuziak et al., 2014a, 2015);
- In (Salman et al., 2014) the strong metric dimension of some convex polytopes is obtained;
- The strong metric dimension of tetrahedral diamond graphs is found in (Rajan et al.);
- The problem of finding the strong metric dimension of circulant graphs and a lower bound for diametrically vertex uniform graphs is solved by Grigorious et al.;
- Some similar dimensions of graphs are introduced in literature: the strong partition dimension (Yero, 2013) and the fractional strong metric dimension (Kang and Yi, 2013).

This paper considers the strong metric dimension of a special class of graphs, so called generalized Petersen graphs. The generalized Petersen graph  $GP(n, k)$  ( $n \geq 3; 1 \leq k < n/2$ ) has  $2n$  vertices and  $3n$  edges, with vertex set  $V = \{u_i, v_i \mid 0 \leq i \leq n-1\}$  and edge set  $E = \{\{u_i, u_{i+1}\}, \{u_i, v_i\}, \{v_i, v_{i+k}\} \mid 0 \leq i \leq n-1\}$ , where vertex indices are taken modulo  $n$ . The Petersen graph from Figure 1 can be considered as  $GP(5, 2)$ .

Generalized Petersen graphs were first studied by Coxeter (1950). Various properties of  $GP(n, k)$  have been recently theoretically investigated in the following areas: metric dimension (Naz et al., 2014), decycling number (Gao et al., 2015), component connectivity (Ferrero and Hanusch, 2014) and acyclic 3-coloring (Zhu et al., 2015).

In the case when  $k = 1$  it is easy to see that  $GP(n, 1) \equiv C_n \square P_2$ , where  $\square$  is the Cartesian product of graphs, while  $C_n$  is the cycle on  $n$  vertices and  $P_2$  is the path with 2 vertices. Using the following result from (Rodríguez-Velázquez et al., 2013):  $sdim(C_n \square P_r) = n$ , for  $r \geq 2$ , it follows that  $sdim(GP(n, 1)) = n$ .

## 2 The strong metric dimension of $GP(n, 2)$

In this section we consider the strong metric dimension of the generalized Petersen graph  $GP(n, 2)$ . The exact value is determined for cases  $n = 4k$  and  $n = 4k + 2$ , while for  $n = 4k + 1$  an upper bound of the strong metric dimension is presented. In order to prove that the sets defined in Lemma 1, Lemma 3 and Lemma 5 are strong resolving we used shortest paths given in Tables 1-3, which is organized as follows:

- First column named "case" contains the case number;
- Next two columns named "vertices" and "res. by" contain a pair of vertices and a vertex which strongly resolves them;
- Last two columns named "condition" and "shortest path" contain the condition under which the vertex in column five strongly resolves the pair in column four and the corresponding shortest path, respectively.

**Lemma 1** Set  $S = \{u_{2i}, v_i \mid i = 0, 1, \dots, 2k\}$  is a strong resolving set of  $GP(4k + 2, 2)$  for  $k \geq 3$ .

**Proof:** Let us consider pairs of vertices such that neither vertex is in  $S$ . There are three possible cases:

- Case 1.  $(u_i, u_j)$ ,  $i, j$  are odd. Without loss of generality we may assume that  $i < j$ . If  $j - i \leq 2k$ , according to Table 1, vertices  $u_i$  and  $u_j$  are strongly resolved by vertex  $u_{i-1}$ . The shortest paths corresponding to the subcases  $j - i = 2$  and  $4 \leq j - i \leq 2k$  are also given in Table 1. As  $i - 1$  is even, then  $u_{i-1} \in S$ . If  $j - i \geq 2k + 2$ , then pair  $(u_i, u_j)$  can be represented as pair  $(u_{i'}, u_{j'})$ , where  $i' = j$  and  $j' = i + 4k + 2$ . Since  $j' - i' \leq 2k$ , the situation is reduced to the previous one.
- Case 2.  $(v_i, v_j)$ ,  $i, j \geq 2k + 1$ . Without loss of generality we may assume that  $i < j$ . If  $j - i$  is even,  $v_i$  and  $v_j$  are strongly resolved by both  $u_j$  and  $u_{j+1}$ . When  $j$  is even  $u_j \in S$ , while  $u_{j+1} \in S$  for odd  $j$ . If  $j - i$  is odd, then  $v_i$  and  $v_j$  are strongly resolved by  $v_{2k} \in S$  if  $i$  is even, and  $v_{2k-1} \in S$  if  $i$  is odd, and hence  $u_i$  and  $u_j$  are strongly resolved by  $S$ ;

**Tab. 1:** Shortest paths in Lemma 1

<i>Case</i>	<i>vertices</i>	<i>res. by</i>	<i>condition</i>	<i>shortest path</i>
1	$(u_i, u_j),$ $i, j$ odd	$u_{i-1}$ $u_{i-1}$	$j - i = 2$ $4 \leq j - i \leq 2k$	$u_{i-1}, u_i, u_{i+1}, u_{i+2} = u_j$ $u_{i-1}, u_i, v_i, v_{i+2}, \dots, v_j, u_j$
2	$(v_i, v_j),$ $i, j \geq 2k + 1$	$u_j$ and $u_{j+1}$ $v_{2k}$ $v_{2k-1}$	$j - i$ even $j - i$ odd, $i$ even $j - i$ odd, $i$ odd	$v_i, v_{i+2}, \dots, v_j, u_j, u_{j+1}$ $v_{2k}, v_{2k+2}, \dots, v_i, u_i, u_{i+1}, v_{i+1}, v_{i+3}, \dots, v_j$ $v_{2k-1}, v_{2k+1}, \dots, v_i, u_i, u_{i+1}, v_{i+1}, v_{i+3}, \dots, v_j$
3	$(u_i, v_j),$ $i$ odd, $j \geq 2k + 1$	$u_{i+1}$ $v_{2k}$ $u_{i-1}$ $u_{i+1}$ $v_{2k-1}$ $u_{j+2} = u_{i+5}$ $u_j$ $u_j$ $v_1$	$i \geq j, j$ odd $i > j, j$ even $j$ odd, $0 < j - i \leq 2k$ $j$ odd, $j - i \geq 2k + 2$ $j - i = 1$ $j - i = 3$ $j$ even, $5 \leq j - i \leq 2k + 1$ $j$ even, $2k + 3 \leq j - i < 4k - 1$ $j$ even, $j - i = 4k - 1$	$v_j, v_{j+2}, \dots, v_j, v_{j+2}, \dots, v_i, u_i, u_{i+1}$ $v_{2k}, v_{2k+2}, \dots, v_j, u_j, u_{j+1}, v_{j+1}, v_{j+3}, \dots, v_i, u_i$ $u_{i-1}, u_i, v_i, v_{i+2}, \dots, v_j$ $v_j, v_{j+2}, \dots, v_i, u_i, u_{i+1}$ $v_{2k-1}, v_{2k+1}, \dots, v_i, u_i, u_{i+1}, v_{i+1} = v_j$ $u_i, u_{i+1}, v_{i+1}, v_{i+3} = v_j, v_{j+2}, u_{j+2}$ $u_i, u_{i+1}, v_{i+1}, v_{i+3}, \dots, v_j, u_j$ $u_j, v_j, v_{j+2}, \dots, v_{i-1}, u_{i-1}, u_i$ $v_j = v_{4k}, v_{j+2} = v_0, u_0, u_1 = u_i, v_1$

Case 3.  $(u_i, v_j)$ ,  $i$  odd,  $j \geq 2k + 1$ . There are nine subcases, characterized by conditions presented in Table 1. Vertices which strongly resolve pair  $(u_i, v_j)$  listed in Table 1 belong to set  $S$ . Indeed, vertices  $v_1, v_{2k-1}$  and  $v_{2k}$  belong to  $S$  by definition, while  $u_{i-1}, u_{i+1}$  and  $u_{i+5}$  belong to  $S$  since  $i$  is odd. Finally,  $u_j \in S$  since  $j$  is even by condition. □

**Lemma 2** *If  $S$  is a strong resolving set of  $GP(4k + 2, 2)$ , then  $|S| \geq 4k + 2$ , for any  $k \geq 2$ .*

**Proof:** Since  $d(u_i, u_{i+2k+1}) = d(v_i, v_{i+2k+1}) = k + 3 = \text{Diam}(GP(4k + 2, 2))$ ,  $i = 0, 1, \dots, 2k$ , from Property 2, at least  $2k + 1$  u-vertices and  $2k + 1$  v-vertices belong to  $S$ . Therefore,  $|S| \geq 4k + 2$ . □

The strong metric dimension of  $GP(4k + 2, 2)$  is given in Theorem 1.

**Theorem 1** *For all  $k$  it holds that  $\text{sdim}(GP(4k + 2, 2)) = 4k + 2$ .*

**Proof:** It follows directly from Lemmas 1 and 2 that  $\text{sdim}(GP(4k + 2, 2)) = 4k + 2$  for  $k \geq 2$ . For  $k = 1$ , using CPLEX solver on ILP formulation (1)-(3), we have proved that set  $S$  from Lemma 1 is also a strong metric basis of  $GP(6, 2)$ , i.e.  $\text{sdim}(GP(6, 2)) = 6$ . □

The strong metric dimension of  $GP(4k, 2)$  will be determined using Lemmas 3 and 4.

**Lemma 3** *Set  $S = \{u_i | i = 0, 1, \dots, 2k - 1\} \cup \{u_{2k+2i+1} | i = 0, 1, \dots, k - 1\} \cup \{v_{2i+1} | i = 0, 1, \dots, 2k - 1\}$  is a strong resolving set of  $GP(4k, 2)$  for  $k \geq 3$ .*

**Proof:** Since  $S$  contains  $u_i$ ,  $i = 1, \dots, 2k - 1$  and all  $u_i$  and  $v_i$  for odd  $i$ , we need to consider only three possible cases:

- Case 1.  $(u_i, u_j)$ ,  $i, j \geq 2k$  are even. Without loss of generality we may assume that  $i < j$ . Vertices  $u_i$  and  $u_j$  are strongly resolved by vertex  $u_{j+1}$  (see Table 2). The shortest paths corresponding to subcases  $j - i = 2$  and  $j - i \geq 4$  are given in Table 2. As  $j + 1$  is odd, then  $u_{j+1} \in S$ ;
- Case 2.  $(v_i, v_j)$ ,  $i, j$  are even. Without loss of generality we may assume that  $i < j$ . If  $j - i \leq 2k - 2$ , vertices  $v_i$  and  $v_j$  are strongly resolved by  $u_{j+1} \in S$ . If  $j - i \geq 2k + 2$ , then pair  $(v_i, v_j)$  can be represented as pair  $(v_{i'}, v_{j'})$ , where  $i' = j$  and  $j' = i + 4k$ . Since  $j' - i' \leq 2k - 2$ , the situation is reduced to the previous one. If  $j - i = 2k$ , vertices  $v_i$  and  $v_j = v_{i+2k}$  are strongly resolved by both  $u_i$  and  $u_j = u_{i+2k}$ . Since  $S$  contains  $u_0, \dots, u_{2k-1}$  it follows that  $u_i$  or  $u_{i+2k}$  belongs to  $S$  and hence  $v_i$  and  $v_j$  are strongly resolved by  $S$ ;
- Case 3.  $(u_i, v_j)$ ,  $i, j$  even,  $i \geq 2k$ . Let us assume first that  $i \leq j$ . If  $j - i \leq 2k - 2$ , vertices  $u_i$  and  $v_j$  are strongly resolved by vertex  $u_{i-1}$ . As  $i - 1$  is odd it follows that  $u_{i-1} \in S$  and  $u_i$  and  $v_j$  are strongly resolved by  $S$ . Assume now that  $i > j$ . If  $j - i = -2k$  then vertices  $u_i$  and  $v_j = v_{i-2k}$  are strongly resolved by vertex  $u_j = u_{i-2k}$ . Since  $S$  contains  $u_0, \dots, u_{2k-1}$  and  $i \geq 2k$ , it follows that  $u_{i-2k}$  belongs to  $S$  and hence  $u_i$  and  $v_j$  are strongly resolved by  $S$ . If  $-2k < j - i < -2$  then pair  $(u_i, v_j)$  is strongly resolved by  $u_{j-1}$ . Since  $j - 1$  is odd, then  $u_{j-1} \in S$ . Finally, if  $j - i = -2$  then pair of vertices  $(u_i, v_j)$  is strongly resolved by  $u_{i+1} \in S$ . These four subcases cover all possible values for  $i$  and  $j$  having in mind that vertex indices are taken modulo  $n$ .

**Tab. 2:** Shortest paths in Lemma 3

<i>Case</i>	<i>vertices</i>	<i>res. by</i>	<i>condition</i>	<i>shortest path</i>
1	$(u_i, u_j),$ $i, j \geq 2k, i, j \text{ even}$	$u_{j+1} = u_{i+3}$ $u_{j+1}$	$j - i = 2$ $j - i \geq 4$	$u_i, u_{i+1}, u_{i+2} = u_j, u_{j+1}$ $u_i, v_i, v_{i+2}, \dots, v_j, u_j, u_{j+1}$
2	$(v_i, v_j),$ $i, j \text{ even}$	$u_{j+1}$ $u_i \text{ and } u_{i+2k}$	$j - i \leq 2k - 2$ $j - i = 2k$	$v_i, v_{i+2}, \dots, v_j, u_j, u_{j+1}$ $u_i, v_i, v_{i+2}, \dots, v_{i+2k} = v_j, u_{i+2k} = u_j$
3	$(u_i, v_j),$ $i, j \text{ even},$ $i \geq 2k$	$u_{i-1}$ $u_{i-2k} = u_j$ $u_{j-1}$ $u_{i+1}$	$0 \leq j - i \leq 2k - 2$ $j - i = -2k$ $-2k < j - i < -2$ $j - i = -2$	$u_{i-1}, u_i, v_i, v_{i+2}, \dots, v_j$ $u_i, v_i, v_{i-2}, \dots, v_{i-2k} = v_j, u_{i-2k} = u_j$ $u_{j-1}, u_j, v_j, v_{j+2}, \dots, v_i, u_i$ $v_j, v_{j+2} = v_i, u_i, u_{i+1}$

□

**Lemma 4** *If  $k \geq 10$  and  $S$  is a strong resolving set of  $GP(4k, 2)$ , then  $|S| \geq 5k$ .*

**Proof:** Let us note that  $d(v_i, v_{i+2k-1}) = k + 2 = \text{Diam}(GP(4k, 2))$ ,  $i = 0, 1, \dots, 4k - 1$ . If we suppose that  $S$  contains less than  $2k$   $v$ -vertices, since we have  $4k$  pairs  $(v_i, v_{i+2k-1})$ ,  $i = 0, \dots, 4k - 1$ , and each vertex appears exactly twice, there exists some pair  $(v_i, v_{i+2k-1})$ ,  $v_i \notin S$ ,  $v_{i+2k-1} \notin S$ . This is in contradiction with Property 2. Therefore,  $S$  contains at least  $2k$   $v$ -vertices.

Considering  $u$ -vertices, we have two cases:

- Case 1. If there exist  $u_{2i} \notin S$  and  $u_{2j+1} \notin S$ , then, as pairs  $\{u_{2i}, u_{2i+2l-1}\}$ ,  $l = 3, 4, \dots, 2k - 2$  and  $\{u_{2j+1}, u_{2j+2l}\}$ ,  $l = 3, 4, \dots, 2k - 2$  are mutually maximally distant, at most 8 additional vertices are not in  $S$ :  $u_{2i-3}, u_{2i-1}, u_{2i+1}, u_{2i+3}, u_{2j-2}, u_{2j}, u_{2j+2}, u_{2j+4}$ . Consequently, at most 10 vertices, where  $10 \leq k$ , are not in  $S$ ;
- Case 2. Indices of  $u$ -vertices which are not in  $S$  are all either even or odd. Without loss of generality we may assume that all these indices are even. Since  $d(u_{2i}, u_{2i+2k}) = k + 2 = \text{Diam}(GP(4k, 2))$ ,  $i = 0, 1, \dots, k - 1$ , according to Property 2, we have  $k$  pairs  $(u_{2i}, u_{2i+2k})$ ,  $i = 0, \dots, k - 1$ , with at most one vertex not in  $S$ . Therefore, at most  $k$   $u$ -vertices are not in  $S$ .

In both cases we have proved that at most  $k$   $u$ -vertices are not in  $S$ , so at least  $3k$   $u$ -vertices are in  $S$ . Since we have already proved that at least  $2k$   $v$ -vertices should be in  $S$ , it follows that  $|S| \geq 5k$ . □

The strong metric dimension of  $GP(4k, 2)$  is given in Theorem 2.

**Theorem 2** *For all  $k \geq 5$  it holds  $\text{sdim}(GP(4k, 2)) = 5k$ .*

**Proof:** Lemma 3 and Lemma 4 imply that  $S = \{u_i | i = 0, 1, \dots, 2k - 1\} \cup \{u_{2k+2i+1} | i = 0, 1, \dots, k - 1\} \cup \{v_{2i+1} | i = 0, 1, \dots, 2k - 1\}$  is a strong metric basis of  $GP(4k, 2)$  for  $k \geq 10$ . Using CPLEX solver on ILP formulation (1)-(3), we have proved that set  $S$  from Lemma 3 is a strong metric basis of  $GP(4k, 2)$  for  $k \in \{5, 6, 7, 8, 9\}$ . □

In the case when  $n = 4k + 1$ , an upper bound of the strong metric dimension of  $GP(4k + 1, 2)$  will be determined as a corollary of the following lemma.

**Lemma 5** *Set  $S = \{u_{2i+1} | i = 0, 1, \dots, k - 1\} \cup \{u_{2k+i} | i = 0, 1, \dots, 2k\} \cup \{v_i | i = 0, 1, \dots, 2k + 3\}$  is a strong resolving set of  $GP(4k + 1, 2)$  for  $k \geq 3$ .*

**Proof:** As in Lemma 1, there are three possible cases:

- Case 1.  $(u_{2i}, u_{2j})$ ,  $0 \leq i, j \leq k - 1$ . Without loss of generality we may assume that  $i < j$ . Vertices  $u_{2i}$  and  $u_{2j}$  are strongly resolved by vertex  $u_{2j+1} \in S$ . The shortest paths corresponding to subcases  $j - i \geq 2$  and  $j - i = 1$  are given in Table 3;
- Case 2.  $(v_i, v_j)$ ,  $2k + 4 \leq i, j \leq 4k$ . Without loss of generality we may assume that  $i < j$ . If  $j$  is even, vertices  $v_i$  and  $v_j$  are strongly resolved by  $v_1 \in S$ , while if  $j$  is odd, they are strongly resolved by  $v_0 \in S$ . The details about shortest paths can be seen in Table 3;



**Tab. 3:** Shortest paths in Lemma 5

<i>Case</i>	<i>vertices</i>	<i>res. by</i>	<i>condition</i>	<i>shortest path</i>
1	$(u_{2i}, u_{2j}),$ $0 \leq i, j \leq k-1$	$u_{2j+1}$ $u_{2j+1}$	$j-i \geq 2$ $j-i = 1$	$u_{2i}, v_{2i}, v_{2i+2}, \dots, v_{2j}, u_{2j}, u_{2j+1}$ $u_{2i}, u_{2i+1}, u_{2i+2} = u_{2j}, u_{2i+3} = u_{2j+1}$
2	$(v_i, v_j),$ $2k+4 \leq i, j \leq 4k$	$v_1$ $v_0$ $v_0$ $v_1$	$i, j$ even $i, j$ odd $i$ even, $j$ odd $i$ odd, $j$ even	$v_i, v_{i+2}, \dots, v_j, v_{j+2}, \dots, v_1$ $v_i, v_{i+2}, \dots, v_j, v_{j+2}, \dots, v_0$ $v_i, u_i, u_{i+1}, v_{i+1}, v_{i+3}, \dots, v_j, v_{j+2}, \dots, v_0$ $v_i, u_i, u_{i+1}, v_{i+1}, v_{i+3}, \dots, v_j, v_{j+2}, \dots, v_1$
3	$(u_{2i}, v_j),$ $0 \leq i \leq k-1,$ $2k+4 \leq j \leq 4k$	$u_j$ $u_j$ $u_j$ $u_j$ $v_{2i}$ $u_1$ $v_0$	$j$ even, $j-2i \leq 2k$ $j$ even, $2k+2 \leq j-2i < 4k-2$ $j$ odd, $j-2i \leq 2k-1$ $j$ odd, $2k+1 \leq j-2i \leq 4k-3$ $j-2i = 4k-2$ $j-2i = 4k-1$ $j-2i = 4k$	$u_{2i}, v_{2i}, v_{2i+2}, \dots, v_j, u_j$ $u_j, v_j, v_{j+2}, \dots, v_{2i-1}, u_{2i-1}, u_{2i}$ $u_{2i}, u_{2i+1}, v_{2i+1}, v_{2i+3}, \dots, v_j, u_j$ $u_j, v_j, v_{j+2}, \dots, v_{2i}, u_{2i}$ $v_j, v_{j+2}, u_{j+2}, u_{j+3}, v_{j+3} = v_{2i}$ $v_j = v_{4k-1}, v_{j+2} = v_0 = v_{2i}, u_0 = u_{2i}, u_1$ $v_j = v_{4k}, u_j = u_{4k}, u_0 = u_{2i}, v_0$

**Tab. 4:** Other strong metric bases of  $GP(n, 2)$  for  $n \leq 19$ 

$n$	$sdim(GP(n, 2))$	$S$
5	8	$\{u_0, u_1, u_2, u_3, v_0, v_1, v_2, v_3\}$
7	9	$\{u_0, u_1, u_2, u_3, v_0, v_1, v_2, v_3, v_4, v_6\}$
8	8	$\{u_4, u_5, u_6, u_7, v_1, v_3, v_5, v_7\}$
9	13	$\{u_2, u_4, u_5, u_6, u_7, u_8, v_0, v_2, v_3, v_4, v_5, v_6, v_7\}$
11	12	$\{u_0, u_1, u_2, u_3, u_4, u_5, v_0, v_1, v_2, v_5, v_6, v_7\}$
12	13	$\{u_0, u_1, u_2, u_3, u_4, u_5, u_6, v_0, v_2, v_4, v_6, v_8, v_{10}\}$
13	17	$\{u_0, u_1, u_2, u_3, u_4, u_5, u_6, u_7\}$ $\cup \{v_1, v_3, v_5, v_6, v_7, v_8, v_9, v_{10}, v_{12}\}$
15	20	$\{u_0, u_1, u_2, u_3, u_4, u_5, u_6, u_7, u_8, u_9\}$ $\cup \{v_0, v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9\}$
16	19	$\{u_0, u_1, u_2, u_3, u_4, u_5, u_6, u_7, u_8, u_9, u_{10}\}$ $\cup \{v_0, v_2, v_4, v_6, v_8, v_{10}, v_{12}, v_{14}\}$
17	24	$\{u_0, u_1, u_2, u_3, u_4, u_5, u_6, u_7, u_8, u_9, u_{10}, u_{11}\}$ $\cup \{v_0, v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9, v_{10}, v_{11}\}$
19	24	$\{u_0, u_4, u_6, u_7, u_8, u_9, u_{10}, u_{11}, u_{12}, u_{13}, u_{14}, u_{15}, u_{16}, u_{17}\}$ $\cup \{v_0, v_1, v_5, v_6, v_9, v_{10}, v_{11}, v_{14}, v_{15}, v_{16}\}$

Case 3.  $(u_{2i}, v_j)$ ,  $0 \leq i \leq k-1$ ,  $2k+4 \leq j \leq 4k$ . In four initial subcases, when  $j-2i < 4k-2$ , vertices  $u_{2i}$  and  $v_j$  are strongly resolved by  $u_j$ . Since set  $S$  contains  $u$ -vertices  $u_{2k}, u_{2k+1}, \dots, u_{4k}$  and  $2k+4 \leq j \leq 4k$  it follows that  $u_j \in S$ . The remaining subcases correspond to situation when  $j-2i \geq 4k-2$ . The pair  $(u_{2i}, v_j)$  is strongly resolved by  $v_{2i}$  for  $j-2i = 4k-2$ , by  $u_1$  for  $j-2i = 4k-1$  and by  $v_0$  for  $j-2i = 4k$ . By definition of  $S$  it is obvious that  $u_1, v_0 \in S$ , while  $v_{2i} \in S$  since  $2i \leq 2k-2$  and all  $v$ -vertices with indices less or equal to  $2k+3$  are in  $S$ . The shortest paths corresponding to these seven subcases can be seen in Table 3.

□

**Corollary 1** If  $k \geq 3$  then  $sdim(GP(4k+1, 2)) \leq 5k+5$ .

The strong metric bases given in Lemma 1 and Lemma 3 and the strong resolving set given in Lemma 5 hold for  $n \geq 20$ . The strong metric bases for  $n \leq 19$  have been obtained by CPLEX solver on ILP formulation (1)-(3). Computational results show that for  $n \in \{6, 10, 14, 18\}$  strong resolving set  $S$  from Lemma 1 is a strong metric basis. The strong metric basis for the remaining cases for  $n \leq 19$  are given in Table 4.

**Open problem 1.** We believe that the strong metric dimension of  $GP(4k+1, 2) = 5k+5$ ,  $k \geq 5$ , but we were not able to give a rigorous proof.

**Open problem 2.** For the remaining case  $n = 4k+3$ ,  $k \geq 5$ , experimental results indicate the following hypothesis:  $sdim(GP(4k+3, 2)) = 5k+6$  for  $k = 5l-2$  and  $sdim(GP(4k+3, 2)) = 5k+4$  for  $k \neq 5l-2$ , where  $l \in \mathbb{N}$ .

### 3 Conclusions

In this paper we have studied the strong metric dimension of generalized Petersen graphs  $GP(n, 2)$ . We have found closed formulas of the strong metric dimensions in cases  $n = 4k$  and  $n = 4k + 2$ , and a tight upper bound of the strong metric dimension for  $n = 4k + 1$ .

A future work could be directed towards obtaining the strong metric dimension of some other challenging classes of graphs.

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